Mathematical Structures in Panini's *ashTAdhyAyI*

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Abstract

To achieve economy in stating the Sanskrit grammar rules for generating conjugal sounds of any two vowels (called *achsandhi*), Panini adopted various techniques that are parallel to some of the modern mathematical techniques. In this article, we describe various artifacts of Panini and demonstrate how Panini built the *achsandhi*-matrix and how his methods resemble the modern mathematical language of algebra.

1 Introduction

Panini's intention behind authoring अष्टाध्यायी (*ashTAdhyAyI*) was to preserve Vedas and the usages of Sanskrit, especially the spoken forms available till

his date (before 500 BC). He aimed to state the grammar rules economically avoiding repetition. In doing so, Panini [1] introduced quite a few structures and operations that can be compared with some well-known and widely used modern mathematical structures such as, sets, sequences, sequence of sequences, equivalent relations, equivalent classes, and actions of a set on another set, etc. Panini could envision these techniques centuries before the conception of modern mathematics. In this article, we took the approach of expressing Panini's methods in current mathematical language to establish these parallels.

In this article, we will present the required material related to the Sanskrit grammar from Panini's *ashTAdhyAyI* [1], [5]. After presenting the mathematical tools required, we will model the material in modern mathematical terms. In doing so, we will be switching between Sanskrit grammar rules and mathematical topics to make the comparisons clear.

Bhate [2] can be of great help in understanding the structure and organization of the content of *ashTAdhyAyI*. A detailed explanation of any Panini's *sUtra* (rule) quoted in this article is available in [1], [5].

We mark important points by naming them as P1, P2, P3 etc and *sUtras* by S1, S2, S3, throughout the article.

Throughout this article, every Panini's *sUtra* is followed by its reference. For example, उच्चैरुदात्तः (1-2-29). The reference (1-2-29) tells us that it is the 29th *sUtra* from the second section of the first chapter.

2 Mathematical Background - Sequences and Sets

Some *mantras* from the early Vedic period (before 2000 BC) invoke powers of ten from a hundred up to a trillion and provide evidence of the use of arithmetic operations such as addition, subtraction, multiplication, fractions etc. Thus, the concept of a numerical sequence was known to Panini. In fact, Panini generalized the idea of a numerical sequence and developed the concept of sequence of letters of Sanskrit.

The modern-day ”sequence” and ”set” in algebra emerged out of many stepwise discoveries in the western mathematical world.

- Fibonacci introduced numerical sequences to the modern mathematics in 1202 AD.
- Set theory was founded and established by Georg Cantor in 1874 AD

In algebra text books, we can find the formal definitions of the above.

Sequence - A sequence is an enumerated collection of objects in which repetitions are allowed.

A finite sequence is denoted by $(a_1, a_2, \ldots\ldots a_n)$ where n is a finite number. Note that we are using the parenthesis $()$ to denote a sequence. For example, (1, 2, 4, 6, 2), (P, A, N, I, N, I), (R, A, M) are sequences.

Each a_i for $i = 1, \ldots, n$ is called **an element (or member or term)** of the sequence and n, the number of elements is called **the length of the sequence**. The same elements can appear multiple times at different positions in a sequence, and order of the elements can not be changed. In a sequence

the enumeration allows us to identify each element specifically.

Set - Naively, a set is supposed to be a collection of 'things' or 'objects' described by 'listing' them or prescribing them by a 'rule' or 'property'. Note that this is not a terribly precise description, but will be adequate for most of our purposes.

We use the parenthesis $\{\}$ to denote a set. For example, {P, A, N, I}, { 1, 2, 4, 6 } are sets.

We can also say that a set is an unordered list of different things. If a thing belongs to a set, we call the thing "an element of the set".

Unlike in a sequence, the elements of a set do not have any particular order and elements are not repeated.

Join of Sequences - If (a, b, c, d) and (a, d, e, f) are two sequences, then their join sequence is (a, b, c, d, a, d, e, f). In mathematical terms, if $A = (a_m)$ and $B = (b_n)$ are two finite sequences, then define

$$c_i = \begin{cases} a_i & \text{for } i = 1, 2, ..m \\ b_{i-m} & \text{for } i = m+1, m+2, ..., m+n \end{cases} \tag{1}$$

then, $C = (c_i)$ is called the join of A and B. Symbolically, we write $C = A \sqcup B$. Note that $A \sqcup B \neq B \sqcup A$.

Truncated Sequence - Suppose k and n are positive integers such that $k < n$ and $(a_1, a_2, \ldots\ldots a_n)$ is a sequence. The sequence $(a_k + 1, a_k + 2, \ldots\ldots a_n)$ formed by removing the first k-elements is called the **left k-truncated sequence** of $(a_1, a_2, \ldots\ldots a_n)$. Let us denote it by $T_k(a_1, a_2, \ldots\ldots a_n)$.

Example: (A, N, I, N, I), (N, I, N, I), (I, N, I), (N, I), (I) are the left 1-, 2-, 3-, 4-, 5- truncated sequences of (P, A, N, I, N, I) respectively.

The Cartesian Product: Let X and Y be any sets. The Cartesian Product X and Y is defined to be the set of all pairs (x, y) where $x \in X$ and $y \in Y$. Symbolically, the Cartesian Product is written as

$$X \times Y = \{(a, b) : x \in X \text{ and } y \in Y\}.$$

The readers are advised to consult any basic book of algebra for the unexplained notation in this article.

In the next sections, we demonstrate how Panini introduced the same concepts in *ashTAdhyAyI* in his own fashion.

2.1 Aphorisms of Shiva

The fourteen *mAheswarasUtraNi* (also known as **Shiva aphorisms**) stated by Panini and elucidated by Patanjali (before 300 BC) [5] are

अइउण् , ऋऌक् , एओङ् , ऐऔच् , हयवरट् , लण् , ञमङणनम् , झभञ् , घढधष् ,
जबगडदश् , खफछठथचटतव् , कपय् , शषसर् , हल्.

To explain these aphorisms and to provide a method to build new aphorisms (called प्रत्याहाराः)), Panini provided some rules.

S1. हलन्त्यम् (1.3.3) – The last consonant letter in each Shiva aphorism is called an **indicatory letter**(इत्). By indicatory, it means that it should be dropped when we consider the aphorism. (The letters other than the indicatory letter

are to be considered and the indicatory letter should be ignored).

S2. अदिरन्त्येन सहेता (1.1.71) – In the above arrangement of fourteen Shiva aphorisms, any combination of two letters (maintaining the order of the letters) with the second letter being a indicatory letter is also an aphorism representing all the significant letters between the first letter and the indicatory letter.

P1. By S1, we know that ण् , क् , ङ् , च् , ट् , म् , ञ् , ष् , श् , व् , य् , र्, ल् are indicatory letters. These letters are used for naming the aphorism only.

For example, अइउण् means only three letters अ, इ, उ and ण् should be ignored.

P2. As the last consonant is indicatory, each Shiva aphorism leads to a sequence of letters.

(अ, इ, उ), (ऋ, ऌ), (ए, ओ), (ऐ औ), (ह य व र), (ल), (ञ, म, ङ, ण, न), (झ, भ), (घ, ढ, ध), (ज, ब, ग, ड, द), (ख, फ, छ, ठ, थ, च,, ट त), (क, प), (श, ष, स), (ह).

The usage of the Shiva aphorisms in *ashTAdhyAyI* clearly indicates that the order of letters in the aphorisms is important and hence they form sequences. We call them **Shiva sequences**.

P3. S2 suggests a scheme for naming the Shiva aphorisms using the first and the last letters. Per that scheme, the names of the Shiva aphorisms (and there by the names of corresponding Shiva sequences) are given by

अण् , ऋक् , एङ् , ऐच् , हट् , लण् , ञम् , झञ् , घष् , जश् , खव् , कय् , शर्, हल् .

Note: From now on, we use aphorism and sequence interchangeably.

P4. The Shiva sequences themselves form a sequence of sequences because the Shiva sequences have an order and the order cannot be changed. Let us call the sequence of the Shiva sequences **SS** where

SS = ((अ, इ, उ), (ऋ, ऌ), (ए, ओ), (ऐ औ), (ह य व र), (ल), (ञ, म, ङ, ण, न), (झ, भ), (घ, ढ, ध), (ज, ब, ग, ड, द), (ख, फ, छ, ठ, थ, च,, ट त), (क, प), (श, ष, स), (ह)).

2.1.1 Generating New Sequences

P5. S2 provides us an algorithm to build new sequences in the following ways:

- Truncating a Shiva sequence.
- Joining two or more Shiva sequences.
- Joining a truncated sequence with two or more Shiva sequences.

P6. Take any Shiva sequence of length m. For each, $k < m$, we can assign a name to its k-left truncated sequence using the naming convention established by S2.

For example, the 1-left truncated sequence of अण् starts with इ and ends with the indicatory letter ण्. So, we call it इण्.

There is a well-known method to build new sequences [1], [5] and we present it here like an algorithm.

Step 1 Start with any significant letter x in any of the Shiva sequences. Suppose x is in k-th position and the indicatory letter of the sequence is s.

Step 2 Consider the $k-1$-truncated sequence of the Shiva sequence starting from x. If $k=1$, it is a non-truncated sequence. Its name will be xs. If this is the sequence of your interest, stop at this step. Else, proceed to the next step.

Step 3 Take the join of the $k-1$-truncated sequence (from Step 2) and its immediate Shiva sequences following the order in **SS** as per your need.

Step 4 Suppose that the indicatory letter of the last Shiva sequence of the join (done in Step 3) is t. Then, the new sequence (formed by the join) will have the name xt.

Note that in Step 3, we are allowed to take joins with only immediate next Shiva sequence. **SS** being a sequence plays a crucial role in this step.

2.2 Examples of Derived Aphorisms:

1. अण् ⊔ ऋक् = अक् = (अ, इ, उ, ऋ, ऌ). Start with अण् in Step 1, choose $k=1$ in Step 2, go with ऋक् in Step 3. Using Step 4, the new sequence is अक्.

2. हट् ⊔ लण् = हण् = (ह, य, व, र, ल). Start with हट् in Step 1, choose $k=1$ in Step 2, go with लण् in Step 3. Using Step 4, the new sequence is हण्.

3. T_1 हण् = T_1 (हट् ⊔ लण्) = यण् = (य, व, र, ल) .

4. अच् = the join of the first four Shiva aphorisms.

5. हल् = the join of the fifth to fourteenth Shiva aphorisms.

6. अल् = the join of all fourteen Shiva aphorisms.

Panini considered only 43 such new sequences in his *ashTAdhyAyI*.

Let us conclude this section with the following observation.

Observation 1: The modern day mathematical notions like algebraic sequence and a sequence of algebraic sequences were established by Panini. He also introduced the unary and binary operations (truncation and join), and a method of generating sequences from a sequence of basic sequences. Panini was centuries ahead in establishing these concepts and operators that are known in the mathematical world now.

3 Mathematical Background - Setoids and Equivalence Classes

Let A be a set. Suppose that there is a property that can relate two elements of A. This property does not need to relate every element with the other. If $a, b \in A$ are related by the property, we say that a **related to** b and represent it symbolically by $a \sim b$. For example, A can be taken as all the cars in a city and the color of the car can be taken as property. Suppose we say that two cars are related if they have the same color. The all red cars are related to each other. A red car and blue car are not related. The property "color" establishes the relation. We can state this concept formally as follows:

A relation $\sim$ is said to be an **equivalence relation** if and only if, for any $a, b, c \in A$, the following are true:

1. $a \sim a$ (Reflexivity)

2. $a \sim b$ implies $b \sim a$ (Symmetry).

3. $a \sim b$ and $b \sim c$ imply that $a \sim c$ (Transitivity).

The pair $(A, \sim)$ is called a **setoid** if $\sim$ is an equivalence relation. (In the above example, "having the same color" is a equivalence relation)

For every $a \in A$, we define the **equivalence class** of a **with respect to the relation** $\sim$, denoted by $[a]$, as

$$[a] = \{x \in A | a \sim x\}.$$

If $(A, \sim)$ is a setoid then A can be decomposed into a disjoint union of equivalent classes of $\sim$. The set of all equivalence classes is denoted by [A] where the relation is understood.

Remarks: Whenever $a \sim b$ then $[a] = [b]$. Also, if a is not related to b their equivalent classes are different. Conceptually, if a is related to b under an equivalence relation, then a and b can be treated as the same with respect to the relation.

In the above example, the cars of the same color form an equivalence class. The number of equivalence classes is the same as the number of various colors of the cars. Even if some cars are removed from the city, the number of equivalence classes remain the same as long as cars of all colors exist.

4 Panini’s Setoids of Vowels

From now on, we will restrict our study to vowels only. First, let us list out the sequences for later use.

Name	Sequence	Set
अक्	(अ, इ, उ, ऋ, ऌ)	{ अ, इ, उ, ऋ, ऌ }
इक्	(इ, उ, ऋ, ऌ)	{ इ, उ, ऋ, ऌ }
एच्	(ए, ओ, ऐ, औ)	{ ए, ओ, ऐ, औ }
अच्	(अ, इ, उ, ऋ, ऌ, ए, ओ, ऐ, औ)	{अ, इ, उ, ऋ, ऌ, ए, ओ, ऐ, औ}

4.1 Length and Tonal Variations of Vowels

Panini introduced additional vowels on the basis of their length and tonal variations. For more information on tonal variations, consult [1].

S3. ऊकालोऽज्झ्रस्वदीर्घप्लुतः (1-2-27) - A vowel whose length is equal to that of उ or ऊ or ऊ3 is called a short (ह्रस्व) or long (दीर्घ) or prolate (प्लुत) vowel respectively.

The time taken for pronouncing a vowel is measured in blinks or snaps. For example, one uses one blink to pronounce उ, two blinks to pronounce ऊ and three blinks to pronounce ऊ3. The number of blinks required to pronounce a vowel is called **the length** of the vowel.

Applying S3 for all vowels, we get a set of 23 vowels with all possible lengths. We call the set अच् * and write it as

अच् * = {अ, आ, अ3, इ, ई, इ3, उ, ऊ, उ3, ऋ, ॠ, ऋ3, ऌ, ॡ, ऌ3, ए, ए3, ओ, ओ3, ऐ, ऐ3, औ, औ3}.

Panini defined the following rules describing the tonal variations of vowels:

S4. उच्चैरुदात्तः (1-2-29) -A letter of अच् * is perceived as having a high tone (acutely accented) is called उदात्तः.

S5. नीचैरनुदात्तः (1-2-30) - A letter of अच् * is perceived as having a low tone (gravely accented) is called अनुदात्तः.

S6. समाहारः स्वरितः (1-2-31) - A letter of अच् * that has the combination of उदात्तः and अनुदात्तः (high and low tones) is called स्वरितः. (circumflexly accented).

P7. For any $v \in$ अच् *, let us denote the स्वरितः, उदात्तः, and अनुदात्तः of v by $v, \overline{v}, \underline{v}$ respectively. With this notation, we can write the set of all 69 vowels with different tones and lengths as अच् ** = $\bigcup_v \{v, \overline{v}, \underline{v}\}$. (Here the union is taken over all $v \in$ अच् *).

Examples: The set of the length and tonal variations of "अ," is {अ, अ॑, अ॒, आ, आ॑, आ॒, अ3, अ॑3, अ॒3 }. For "ए,", it is { ए, ए॑, ए॒, ए3, ए॑3, ए॒3 }.

Each one of the above 69 vowels of अच् ** has two more variations called अनुनासिकः (nasal) and अननुनासिकः (non-nasal). We will restrict our attention to non-nasal only.

4.2 An Equivalence Relation on अच्* and अच्**

In Sanskrit, the sounds of letters are produced from the throat, inner jaws, the roof of palate, teeth, and lips, etc. These places are called **the origins of pronunciation** or simply origins. Moreover, to pronounce a letter, some **effort** is needed [1]. In other words, when a letter is uttered, it is a cumulative

effect of the effort and the origin of pronunciation.

It is observed that a vowel and its variations have the same the effort and the origin of pronunciation. Panini captured this homogeneity of pronunciation as property in the following rule.

S7. तुल्यास्यप्रयत्नम् सवर्णम् (1-1-9) - If any two letters require the same origin of pronunciation and effort to pronounce them, then one is said to be सवर्णम् (**homogeneous**) to the other.

Example: To pronounce अ and आ, the required origins and efforts are the same. Therefore, अ is homogeneous to all its variations. Clearly, अ is not homogeneous to इ because the origins of pronunciations are different.

The "homogeneous" property is a relation on अच् * and अच् **.

It is easy to verify that "homogeneous" relation (or **homogeneity**) is reflexive, symmetric and transitive.

P8. The "homogeneity" is an equivalence relation on अच्** and अच्*. Let us denote the homogeneous relation by a symbol $\sim$. We can conclude that (अच्**, $\sim$) and (अच्*, $\sim$) are setoids.

Note: The number of elements in (अच्**, $\sim$) and (अच्*, $\sim$) are the same.

4.3 Equivalence Classes of Homogeneity Relation

To identify the equivalence classes, we need to inspect the origin and effort to pronounce vowels. First, let us list out the origins of pronunciation of the elements of अच्** below.

Letters	Origin
अ, अ॑, अ॒, आ, आ॑, आ॒, अ3, अ॑3, अ॒3	Throat
इ, इ॑, इ॒, ई, ई॑, ई॒, इ3, इ॑3, इ॒3	Inner Jaw
ऋ, ऋ॑, ऋ॒, ॠ, ॠ॑, ॠ॒, ऋ3, ऋ॑3, ऋ॒3	Head/Roof of the Mouth
ऌ, ऌ॑, ऌ॒, ॡ, ॡ॑, ॡ॒, ऌ3, ऌ॑3, ऌ॒3	Teeth
उ, उ॑, उ॒, ऊ, ऊ॑, ऊ॒, उ3, उ॑3, उ॒3	Lips
ए, ए॑, ए॒, ए3, ए॑3, ए॒3, ऐ, ऐ॑, ऐ॒, ऐ3, ऐ॑3, ऐ॒3	Throat and Inner Jaw
ओ,ओ॑, ओ॒, ओ3,ओ॑3, ओ॒3, औ, औ॑, औ॒, औ3, औ॑3, औ॒3	Throat and Lips

Remark 1: Each row of the left hand side column has its own origin of pronunciation. So, there is no homogeneity between the elements of one row and the other row.

Remark 2: Due the effort differences, ए and ऐ are **not** homogeneous to each other and ओ and औ are **not** homogeneous to each other.

With the help of above remarks, we can write down the set of all the equivalence classes of अच्** with respect $\sim$.

[अच्**] = {[अ], [इ], [उ], [ऋ], [ऌ], [ए], [ओ], [ऐ], [औ] }

where

[अ] = {अ, अ॑, अ॒, आ, आ॑, आ॒, अ3, अ॑3, अ॒3},

[इ] = {इ, इ॑, इ॒, ई, ई॑, ई॒, इ3, इ॑3, इ॒3},

[उ] = {उ, उ॑, उ॒, ऊ, ऊ॑, ऊ॒, उ3, उ॑3, उ॒3},

[ऌ] = {ऌ, ऌ॑, ऌ॒, ॡ, ॡ॑, ॡ॒, ऌ3, ऌ॑3, ऌ॒3},

[ए] = {ए, ए॑, ए॒, ए3, ए॑3, ए॒3},

[ऋ] = {ऋ, ऋ॑, ऋ॒, ॠ, ॠ॑, ॠ॒, ऋ3, ऋ॑3, ऋ॒3},

[ओ] = {ओ, ओ॑, ओ॒, ओ3,ओ॑3, ओ॒3},

[ऐ] = {ऐ, ऐ॑, ऐ॒, ऐ3, ऐ॑3, ऐ॒3},

[औ] = {औ, औ॑, औ॒, औ3, औ॑3, औ॒3}.

Similarly, the equivalence classes of अच्* with respect $\sim$ are given by

$$[\text{अच्*}] = \{ [\text{अ}], [\text{इ}], [\text{उ}], [\text{ऋ}], [\text{ऌ}], [\text{ए}], [\text{ओ}], [\text{ऐ}], [\text{औ}] \}$$

where

[अ] = {अ, आ, अ3},

[इ] = {इ, ई, इ3},

[उ] = {उ, ऊ, उ3},

[ऋ] = {ऋ, ॠ, ऋ3},

[ऌ] = {ऌ, ॡ, ऌ3 },

[ए] = {ए, ए3},

[ओ] = {ओ, ओ3},

[ऐ] = {ऐ, ऐ3},

[औ] = {औ, औ3}.

We conclude this section by stating the following important observations.

Observation 2: Panini identified a phonetic relation that can be utilized to split the set of all vowels and its variations into 9 disjoint sets of vowels. In each set, the vowels are homogeneous to each other. This kind of classification is exactly parallel to the modern-day concept of dividing a set into disjoint equivalence classes based on an equivalence relation.

Commentators of *ashTAdhyAyI* stated a very important fact about the अच्–

सन्धिः rules.

Observation 3: When Panini stated the first four Shiva aphorisms, he meant the equivalence classes of the vowels, not just vowels. This means, अण्, ऋक्, एङ्, ऐच् are in fact ([अ], [इ], [उ]), ([ऋ], [ऌ]), ([ए], [ओ]), ([ऐ], [औ]) as sequences.

In the next section, we choose the equivalence class with length variations only and study how Panini used some specific equivalence classes to describe *achsandhi.*

P9. In the rest of the article, we consider the equivalence classes with length variations only. We list out a few sets of equivalence classes for future use.

1. [अच्*] = {[अ], [इ], [उ], [ऋ], [ऌ], [ए], [ओ], [ऐ], [औ]}.
2. [अक्*] = { [अ], [इ], [उ], [ऋ]}.
3. [इक्*] = {[इ], [उ], [ऋ], [ऌ]}.
4. [एच्*] = {[ए], [ओ], [ऐ], [औ]}.
5. We consider the complements of equivalence classes in our study, We list them too.
   $[इ]^c$ in [अच्*] = { [अ], [उ], [ऋ], [ऌ], [ए], [ओ], [ऐ], [औ] }
   $[उ]^c$ in [अच्*] = { [अ], [इ], [ऋ], [ऌ], [ए], [ओ], [ऐ], [औ] }
   $[ऋ]^c$ in [अच्*] = { [अ], [इ], [उ], [ऌ], [ए], [ओ], [ऐ], [औ] }
   $[ऌ]^c$ in [अच्*] = { [अ], [इ], [उ], [ऋ], [ए], [ओ], [ऐ], [औ] }.

5 अच्–सन्धिः(*achsandhi*) Rules - Panini's Substitution Mappings

Panini formally defined the closest proximity of any two letters. When two vowels are in the closest proximity, the vowels undergo changes to form a conjugate. The rules that describe these changes are called अच्–सन्धिः rules.

S8. परस्संनिकर्षस्संहिता (1-4-109) - Extreme closeness between two letters is called संहिता (*closest proximity*).

सन्धिः(*sandhi*) operation is possible when there is संहिता (closest proximity) between two letters. We use "+" to represent the *sandhi* operation.

For example, consider सुधी + उपास्यः. There is संहिता (closest proximity) between ई in धी and उ , सन्धिः happens to ई and उ. The *sUtra* इको यणचि (6-1-77) will decide the resultant conjugate.

We can expect 529 *sandhi*-conjugates (formed by 23 vowels of [अच्*] with themselves) specified in the *achsandhi* rules. Indeed, Panini covers all 529 conjugate descriptions only in 5 rules by using the 9 equivalence classes of [अच्*].

The purpose of this section is

1. to find substitution mappings that describe *achsandhi* rules for vowels.
2. to demonstrate the extensive use of equivalence classes in formulating *achsandhi* rules.

5.1 Mathematical Background for अच्–सन्धिः

Complement of a Set - Let M be a subset of a set U. We define **the complement** of M in U as the set of all elements in U but not in M. This set is denoted by M^c.

Let S and T be two sets. By a **mapping** from S to T, we mean an assignment process ψ in which for every $s \in S$ we assign a unique $t \in T$. We write this process symbolically as $\psi : S \to T$ and the assignment as $\psi(s) = t$.

We consider the following sets and notation in the rest of the article: $A = \{a_1, a_2, a_3, \ldots, a_m\}$ and $C = \{c_1, c_2, c_3, \ldots, c_m\}$ are finite sets. X is a finite set and $\sim$ is an equivalent relation on X. $[X] = \{[x_1], [x_2], \ldots [x_m]\}$ is the set of equivalence classes. For every $x \in X$, there exist an $[x_i] \in [X]$ such that $[x] = [x_i]$.

Given any two vowels u and v, suppose an अच्–सन्धिः declares that $u+v = w$ where w is a letter. This अच्–सन्धिः expression $u + v = w$ can be understood as a mapping that assigns (u, v) to w. We call it **”substitution mapping”**. For example, अ + अ = आ as per *achsandhi* rule. The same expression can be written as the substitution mapping (अ, अ) $\to$ आ.

5.2 यणादेशसन्धिः (*yaNAdeSasandhi*) – A Family of Constant Mappings on the Complement Domains

5.2.1 Mathematical Mappings for यणादेशसन्धिः

Suppose there exists a constant mappings $\omega([x_i]) : [x_i]^c \rightarrow c_i$ for all $i = 1, 2, \ldots, m$.

The assignment $\omega : [x_i] \rightarrow \omega([x_i])$ is called **a family of constant mappings on the complement domains** indexed by the set $[X]$.

Each $\omega([x_i])$ induces a bigger family of maps $\tilde{\omega}(x) : [x]^c \rightarrow C$ defined by $\tilde{\omega}(x)(v) = \omega([x])(v)$ for $x \in X$ and $v \in [x]^c$. ($\tilde{\omega}$ is well-defined because if $x = y$ then $[x] = [y]$). We may view this $\tilde{\omega}(x)(v)$ as $\tilde{\omega}(x, v)$.

5.2.2 यणादेशसन्धिः

S9. इको यणचि (6-1-77) - The consonants of यण् = { य्, व्, र्, ल् } are the substitutes of corresponding vowels in इक् when followed by a vowel that is not homogeneous to them.

Let us set X = इक्, C = यण् = {य्, व्, र्, ल् } and build the above mathematical mappings.

ω, the family of constant mappings on the complement domains indexed by the elements of_[इक्*] into यण् can be written as

1. ω([इ]): [इ]$^c \rightarrow$ य् ;

2. ω([उ]): [उ]$^c \rightarrow$ व् ,

3. ω([ऋ]): $[\text{ऋ}]^c \to$ र् ,

4. ω([ऌ]): $[\text{ऌ}]^c \to$ ल् .

The induced mappings at each individual vowel level are:

1. $\tilde{\omega}(v, w) \to$य् + w for $v \in [\text{इ}]$ and $w \in [\text{इ}]^c$

2. $\tilde{\omega}(v, w) \to$व् + w for $v \in [\text{उ}]$ and $w \in [\text{उ}]^c$,

3. $\tilde{\omega}(v, w) \to$र्+ w for $v \in [\text{ऋ}]$ and $w \in [\text{ऋ}]^c$,

4. $\tilde{\omega}(v, w) \to$ल् + w for $v \in [\text{ऌ}]$ and $w \in [\text{ऌ}]^c$.

These are the substitution mappings for the "यणादेशसन्धिः" (ref. S9). To understand in more detail, let us expand up on the first mapping. It says that if इ or ई or इ3 is in closest proximity to any vowel of $[\text{इ}]^c$ then v will be replaced by य् .

5.3 सवर्णदीर्घसन्धिः (*savarNa-dIrghasandhi*) – A Mapping on the Diagonal

5.3.1 Mathematical Mappings for सवर्णदीर्घसन्धिः

The set defined by
$D = \{([x_1], [x_1]), ([x_2], [x_2]), \ldots, ([x_m], [x_m])\} \subset [X] \times [X]$
is called the **diagonal** of $[X]$.

The mapping $\sigma : D \to C$ defined by $\sigma(([x_i], [x_i])) = c_i$ for $i = 1, 2, \ldots, m$ is called **a mapping on the diagonal of** $[X]$.

Now, define a set $\tilde{D^i} = \{(t, s) : t, s \in [x_i])\}$ for each i.

σ induces mappings $\sigma^i : \tilde{D^i} \to C$ given by $\tilde{\sigma}(t, s) = c_i$ where $(t, s) \in D^i$.

5.3.2 सवर्णदीर्घसन्धिः

S10. अकस्सवर्णे दीर्घः (6-1-99) – When a vowel in अक् is followed by a vowel that is homogeneous to it, the corresponding long vowel is the single substitute for both the precedent and the subsequent vowels.

Let us set X = अक्*, C = {आ, ई, ऊ, ॠ }. and build the above mathematical mappings. Then, $D = \{$ ([अ], [अ]), ([इ], [इ]), ([उ], [उ]), ([ऋ], [ऋ])}. and the diagonal mapping $\sigma : D \to$ {आ, ई, ऊ, ॠ } defined by

$$\sigma([\text{अ}], [\text{अ}]) \to \text{आ}; \quad \sigma([\text{इ}], [\text{इ}]) \to \text{ई};$$
$$\sigma([\text{उ}], [\text{उ}]) \to \text{ऊ}; \quad \sigma([\text{ऋ}], [\text{ऋ}]) \to \text{ॠ};$$

induces the four mappings

$$\sigma^1(u, v) \to \text{आ for any } v, w \in [\text{अ}]$$
$$\sigma^2(u, v) \to \text{ई for any } v, w \in [\text{इ}],$$
$$\sigma^3(u, v) \to \text{ऊ for any } v, w \in [\text{उ}]$$
$$\sigma^4(u, v) \to \text{ॠ for any } v, w \in [\text{ऋ}].$$

These induced mappings σ^i give the substitution mappings for the ”(सवर्णदी-र्घसन्धिः)” (ref. S10). Grammarians declared that [ऋ]=[ऌ] [1], [5].

5.4 गुणसन्धिः(*guNasandhi*) – An Action Mapping

5.4.1 Mathematical Mappings for गुणसन्धिः

Consider the mapping $\gamma : A \times [X] \rightarrow C$ defined by $\gamma(a)([x_i]) = c_i$ for all $i = 1, 2, \ldots, m$ and $a \in A$. It is called **action** of A on $[X]$ onto C.

This action induces $\tilde{\gamma} : A \times X \rightarrow C$ defined by $\tilde{\gamma}(a, w) = \gamma(a)([w])$ for any $a \in A$ and $w \in X$. Recall that for any $w \in X$, there exists an $[x_i]$ such that $[w] = [x_i]$.

5.4.2 गुणसन्धिः

S11. आद्गुणः (6-1-87) - The *guNa* is the single substitute of the final अ or आ of a preceding word and the simple vowel of the succeeding.

By definition, the *guNa* substitute set contains {ए, ओ, अर्, अल्} [1].

Let us set X = इक्*, A = {अ, आ}, C = {ए, ओ, अर्, अल् } and build the above mathematical mappings.

The action map γ: {[अ]} × [इक्*] → {ए, ओ, अर्, अल्} is defined by

γ (a, [इ])) = {ए}, γ (a, [उ])) = {ओ},

γ (a, [ऋ])) = {अर्}, γ (a, [ऌ])) = {अल्}

for any a ∈ {अ, आ}. The induced maps are given by

$\tilde{\gamma}(a, e) \rightarrow$ ए for any a ∈ {अ, आ}, and $e \in$ [इ],

$\tilde{\gamma}(a, u) \rightarrow$ ओ for any a ∈ {अ, आ}, and $u \in$ [उ]

$\tilde{\gamma}(a, r) \to$ अर् for any a $\in$ {अ, आ}, and $u \in$ [ऋ]

$\tilde{\gamma}(a, l) \to$ अल् for any a $\in$ {अ, आ}, and $u \in$ [ऌ]

provide the substitution mappings for ”गुणसन्धिः” (ref. S11).

5.5 वृद्धिसन्धिः(*vRiddhisandhi*)– An Action Mapping

The mathematical mapping used for गुणसन्धिः works for वृद्धिसन्धिः also.

5.5.1 वृद्धिसन्धिः

S12. वृद्धिरेचि (6-1-88) - The *vRiddhi* is the single substitute of अ or आ of a preceding word and in the initial diphthong the succeeding vowel.

By definition, the *vRiddhi* substitute set contains {ऐ, औ} [1].

Let us set X = एच्*, A = {अ, आ}, C = {ऐ, औ } and build the above mathematical mappings.

The action map ρ: {अ, आ} × [एच्*] $\to$ {ऐ, औ} is defined by

ρ(a , [ए]) = {ऐ}, ρ (a, [ऐ]) = {ऐ},

ρ(a, [ओ]) = {औ} ρ (a, [औ]) = {औ}

for any a $\in$ {अ, आ}. The induced maps given by

$\tilde{\rho}(a, e) \to$ {ऐ} for any a $\in$ [अ] and $e \in$ [ए],

$\tilde{\rho}(a, e) \to$ {ऐ} for any a $\in$ [अ] and $e \in$ [ऐ],

$\tilde{\rho}(a, o) \to$ {औ} for any a $\in$ [अ] and $o \in$ [ओ],

$\tilde{\rho}(a, u) \to$ {औ} for any a $\in$ [अ] and $u \in$ [औ]

provide the substitution mappings for "वृद्धिसन्धिः" (ref. S12).

5.6 यान्तवान्तादेशसन्धिः (*yAntAvAntAdeSasandhi*) – An Action by Constant Mappings

5.6.1 Mathematical Mappings for यान्तवान्तादेशसन्धिः

Let V, W be two sets and $\sim$ is an equivalence relation on both of them. Let U be another set such that $[V]$ and U have the same number of elements. Suppose that for every $[v] \in [V]$, there exists a constant mapping $\eta([v]) : W \to u$ for $u \in U$. We call the assignment $\eta : v \to \eta([v])$, an **action by constant mappings** on W (indexed by the set $[V]$) into U.

η induces a mapping $\tilde{\eta} : V \times W \to U$ given by $\tilde{\eta}(v, w) = \eta([v])(w)$.

5.6.2 यान्तवान्तादेशसन्धिः

S13. एचोऽयवायावः (6-1-78) - When a vowel from अच् follows ऐच्, then {अय्, आय्, अव्, आव्} must be substituted in place of ऐच् respectively .

Let us set V = एच्*, W = अच्*, U = {अय्, आय्, अव्, आव् } and build the above mathematical mappings.

The action by constant mappings η: [एच्*]] $\times$ अच्* $\to$ {अय्, आय्, अव्, आव्} defined by

η([ए], v) = अय् + v, η([ऐ], v) = आय् + v,

$$\eta([\text{ओ}], v) = \text{अव्} + v, \qquad \eta([\text{औ}], v) = \text{आव्} + v$$

for any $v \in$ अच्*. Then, the induced mappings given by

$$\tilde{\eta}(a, v) \to \text{अय्} + v \qquad \text{for any } a \in [\text{ए}],$$
$$\tilde{\eta}(b, v) \to \text{आय्} + v \qquad \text{for any } b \in [\text{ऐ}],$$
$$\tilde{\eta}(c, v) \to \text{अव्} + v \qquad \text{for any } c \in [\text{ओ}],$$
$$\tilde{\eta}(d, v) \to \text{आव्} + v \qquad \text{for any } d \in [\text{औ}]$$

for any $v \in$ अच्*, $\tilde{\eta}$ provides the substitution mappings for "यान्तवान्तादेशसन्धिः" described in S13.

5.7 Summary

We present all the above *sandhi* maps in a nutshell below in a matrix of the equivalence classes with substitute mappings as entries.

<table>
<tr><td></td><td>[अ]</td><td>[इ]</td><td>[उ]</td><td>[ऋ]</td><td>[ऌ]</td><td>[ए]</td><td>[ऐ]</td><td>[ओ]</td><td>[औ]</td></tr>
<tr><td>[अ]</td><td>σ^1</td><td colspan="4">$\tilde{\gamma}$</td><td colspan="4">$\tilde{\rho}$</td></tr>
<tr><td>[इ]</td><td>$\tilde{\omega}$</td><td>σ^2</td><td colspan="7">$\tilde{\omega}$</td></tr>
<tr><td>[उ]</td><td colspan="2">$\tilde{\omega}$</td><td>σ^3</td><td colspan="6">$\tilde{\omega}$</td></tr>
<tr><td>[ऋ]</td><td colspan="3" rowspan="2">$\tilde{\omega}$</td><td colspan="2" rowspan="2">σ^4</td><td colspan="4" rowspan="2">$\tilde{\omega}$</td></tr>
<tr><td>[ऌ]</td></tr>
<tr><td>[ए]</td><td colspan="9" rowspan="4">$\tilde{\eta}$</td></tr>
<tr><td>[ऐ]</td></tr>
<tr><td>[ओ]</td></tr>
<tr><td>[औ]</td></tr>
</table>

In order to simplify the above matrix, let us drop the prolate vowels in the summary as they are not very frequent in the language. We can draw a matrix with vowel entries as follows:

<table>
<tr><td></td><td>अ</td><td>आ</td><td>इ</td><td>ई</td><td>उ</td><td>ऊ</td><td>ऋ</td><td>ॠ</td><td>ऌ</td><td>ए</td><td>ऐ</td><td>ओ</td><td>औ</td></tr>
<tr><td>अ</td><td colspan="2" rowspan="2">σ^1</td><td colspan="7" rowspan="2">$\tilde{\gamma}$</td><td colspan="4" rowspan="2">$\tilde{\rho}$</td></tr>
<tr><td>आ</td></tr>
<tr><td>इ</td><td colspan="2" rowspan="2">$\tilde{\omega}$</td><td colspan="2" rowspan="2">σ^2</td><td colspan="9" rowspan="2">$\tilde{\omega}$</td></tr>
<tr><td>ई</td></tr>
<tr><td>उ</td><td colspan="4" rowspan="2">$\tilde{\omega}$</td><td colspan="2" rowspan="2">σ^3</td><td colspan="7" rowspan="2">$\tilde{\omega}$</td></tr>
<tr><td>ऊ</td></tr>
<tr><td>ऋ</td><td colspan="6" rowspan="3">$\tilde{\omega}$</td><td colspan="2" rowspan="3">σ^4</td><td colspan="5" rowspan="3">$\tilde{\omega}$</td></tr>
<tr><td>ॠ</td></tr>
<tr><td>ऌ</td></tr>
<tr><td>ए</td><td colspan="13" rowspan="4">$\tilde{\eta}$</td></tr>
<tr><td>ऐ</td></tr>
<tr><td>ओ</td></tr>
<tr><td>औ</td></tr>
</table>

It is clear from the table that Panini defines a well-defined $sandhi$ operation as a map

Φ:अच्*$\times$अच्*$\rightarrow${आ, ई, ऊ, ॠ, ए, ऐ, ओ, औ, य्,व्,र्,ल्,अर्, अय्, अव्, आय्, आव् }

by combining various mappings as shown in the above table. This mapping Φ provides the substitution rules for अच्–सन्धिः.

6 Conclusion

Panini's method followed the following steps.

Step 1 - He reduced the problem of describing 23 X 23 matrix into describing 9X9 matrix.

Step 2 - He decomposed the 9X9 matrix into 13 sub-matrices, that can described by five substitution mappings.

Step 3 - He defined five rules describing the five substitution mappings.

Step 4 - His substitution mappings are defined on the equivalence classes and then lifted to the total set of vowels

Step 5 - He finally defined an action of a set on itself (Φ in the current case) by properly weaving mappings and actions on sets.

Observation 4: Utilizing the equivalence classes of homogeneity combined with the derived sequences ingeniously, Panini could describe the 81 combinations in just five rules.

Observation 5: Panini's scheme of designing *achsandhi* rules establishing the required artifacts and logically chaining them to get the final results is not just a coincidence or chance. All the above procedural steps demonstrate his "economy of expression" and logical thinking that matches with current-day mathematical thinking and algorithmic approach.

It is easy to state rules of other *sandhi*-s (such as *halsandhi*) in a similar way by building relevant mathematical structures.

Panini came up with a plethora of ideas to organize the known grammatical forms of his day systematically. Like a mathematician who models a known phenomenon in mathematical language, Panini created a metalanguage and it

is very close to the modern-day ideas of algebra. He had, though not in the modern terms, a logical understanding of concepts like order sets, sequences, equivalence relations and their equivalence classes, compliments of sets, set relations, the action of a set on another set by well-defined mappings, etc.

For Panini, "Order" is the most important instrument in organizing his material in *ashTAdhyAyI* to obtain precision and accuracy. He arranged his 3981 rules in such a way that the order dictates the 'information chaining'. For example, to get the complete meaning of a rule, often the readers need to borrow some words from the previous rules. This practice is called अनुवृत्ति (Ellipsis). This clearly shows the set of rules in *ashTAdhyAyI* is a sequence with no element repeated.

There are many ingenious ways in which the grammatical information was presented through various structures in *ashTAdhyAyI*. No wonder, Patanjali rightly describes Panini as a person of infinite wisdom (अनल्पमतिः).

The purpose of this article is to expose a few logical ideas of Panini and expose their parallels developed in the later years in modern mathematics. We launched a systematic study to elucidate the Paninian structures in modern terms through a series of articles. For example, in a separate article, we present the mathematical structures parallel to Panini's methods to build Sanskrit vocabulary from basic words such as the roots of nouns and adjectives, etc. and generate declensions [6].

References

[1] S.C. Vasu. *"Ashtadhyayi of Panini" 8 volumes*. Motilal Banarasidas, India.1891.

[2] Saroja Bhate *Panini (Makers of Indian Literature)*. Sahitya Academy, India. 2002.

[3] L.E. Sigler *Fibonacci's Liber Abaci (English translation)*. Springer-Verlag. 2002.

[4] Georg Cantor *Ueber eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen*. Journal für die reine und angewandte Mathematik (in German), 77: 258–262. 1874.

[5] Pandit Nandakishore Sastry (Editor) *Patanjali's Vyakarana Mahabhashya*. Pratibha Prakashan, Delhi, India.1938.

[6] Bhaskar Kompella *"Factory Model for Sanskrit Vocabulary Generation"* (Preprint)